Nonlinear Schrödinger equation in nematic liquid crystals

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We derive the amplitude equation, in the weakly nonlinear regime, for an optical wave packet that propagates in an initially undistorted nematic liquid crystal. By using the dyad representation Q_{ij}, we find the retarded and nonlocal equation for the nematic configuration and solve it in Fourier space. This allows us to calculate the amplitude dependent dispersion relation for a nematic liquid crystal in a given initial undistorted stationary state. We consider a linearly polarized wave packet that travels along the principal axis of the nematic dielectric tensor. We find a nonlinear Schrödinger equation for the amplitude, which includes an additional quadratic term with dissipation.

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I. INTRODUCTION

In recent years there has been a great deal of interest in the nonlinear optics of liquid crystals because of the giant optical nonlinearity of these materials (a factor of 6–10 orders of magnitude larger [1] than that of doped glasses) and the strong nonlinear effects [2] that can be achieved in nematic liquid crystals by using laser with moderate intensity (kW/cm^2). Some recent experiments [3] using continuous beams have shown the presence of steady spatial patterns for cylindrical [3] and planar [4] geometries. The basic mechanism that governs these time-independent patterns is the balance between the nonlinear refraction (self-focusing) and the spatial diffraction of the nematic liquid crystal. A study of these experiments using separation of scales [5,6] shows that the field amplitude at the center of a Gaussian beam (inner solution) follows a nonlocal nonlinear Schrödinger (NLS) equation that is able to describe the undulation and filametation observed in the experiments.

A distinct result is obtained when the propagation of wave packets instead of continuous beams is considered. In this case there exists the possibility of stable and robust solitary wave solutions (optical solitons) when the equilibrium between dispersion and self-focusing is reached. This possibility for a planar waveguide in a specific distorted configuration has been previously considered [7]; however, the lossy and nonlocal aspects of the director dynamics were neglected.

The aim of the present work is to take into account the time dependence and nonlocal effects both in the light and in the nematic liquid crystal by using the dyad representation Q_{ij} of the nematic orientation in describing optical pulses that travel in a nematic material. One of the advantages of the dyad representation is that in the weakly nonlinear regime the relation between Q_{ij} and the electric field dyad is linear and as a consequence the nonlocal effects can be readily incorporated.

The outline of the paper is as follows. In Sec. II we derive from the Hemlholtz free energy the dynamical equation for the dyad Q_{ij} in terms of the nonlocal polarization vector. In Sec. III we express Q_{ij} in terms of the electric field for a weakly nonlinear regime and find a closed nonlocal and retarded equation for the electric field. In Sec. IV, we consider the amplitude equation for a linearly polarized narrow-band wave packet and show that it is the NLS equation with an additional complex quadratic term. In Sec. V we calculate characteristic parameters of the soliton and compare them with those in glass fibers. In Sec. VI we summarize our results.

II. BASIC EQUATIONS

The Hemholtz free energy for a nematic liquid crystal in the presence of an external optical field E_α can be written as

$$F=\int_V dV\mathcal{F}, \tag{2.1}$$

where the energy density $\mathcal{F}$ is given by [8]

$$\mathcal{F}=\frac{L_1}{2}\partial_\beta Q_{\alpha\gamma}\partial_\beta Q_{\alpha\gamma}+\frac{L_2}{2}\partial_\alpha Q_{\alpha\gamma}\partial_\beta Q_{\beta\gamma}+\frac{L_3}{2}\partial_\alpha Q_{\beta\gamma}\partial_\gamma Q_{\beta\alpha} -\frac{D_\mu E_\mu^*+D_\mu^* E_\mu}{2}-\frac{Q_{\alpha\beta}}{2}(D_\alpha^a E_\beta^*+D_\alpha^{a*}E_\beta), \tag{2.2}$$

where L_1, L_2, and L_3 are elastic constants, $Q_{\mu\nu}=n_\mu n_\nu$, with n_μ the nematic director, and D_α, D_α^a are displacement vectors defined by the nonlocal and retarded relations

$$D_\alpha(\vec{r},t)=\int d\vec{r}\,'^3dt'\,\epsilon_\perp(\vec{r}-\vec{r}\,',t-t')E_\alpha(\vec{r}\,',t') \tag{2.3}$$

and

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$$D^a_\alpha(\vec{r},t)=\int d\vec{r}'^3 dt' \epsilon_a(\vec{r}-\vec{r}',t-t')E_\alpha(\vec{r}',t'). \quad (2.4)$$

Equation (2.2) assumes that the degree of orientational order, usually described by the order parameter S, is constant. This is a reasonable approximation for temperatures far from the nematic-isotropic transition and for modest fields.

To obtain the equation of motion it is necessary to describe the generalized thermodynamic force acting on the dyad $Q_{\beta\gamma}$. For this purpose we calculate the variation of $\mathcal{F}$ in Eq. (2.2) with respect to $Q_{\alpha\beta}$, with the constraints that $Q_{\beta\gamma}$ is idempotent and symmetric, which follows from $\hat{n}\cdot\hat{n}=1$. In the spirit of Ref. [9], we write the free energy in terms of the nonsymmetric and nonidempotent tensor $Q_{\alpha\beta}=Q^0_{\alpha\beta}+Q^0_{\beta\alpha}-(Q^0_{\alpha\gamma}Q^0_{\gamma\beta}+Q^0_{\beta\alpha}Q^0_{\alpha\gamma})/2$ and consider variations $\sigma_{\mu\alpha}$ of the tensor $Q^0_{\beta\lambda}(\vec{r})$ such that $\sigma_{\mu\lambda}$ vanishes on the sample boundaries. In this way we obtain

$$Q_{\lambda\beta}=Q^0_{\alpha\beta}+[(\delta_{\lambda\alpha}-Q^0_{\lambda\alpha})\delta_{\beta\mu}+(\delta_{\beta\alpha}-Q^0_{\beta\alpha})\delta_{\lambda\mu}]\sigma_{\mu\alpha}, \quad (2.5)$$

where $Q^0_{\lambda\beta}$ is an arbitrary tensor about which variations are considered. The term in the square brackets is a projection operator that projects out parts of $\sigma_{\mu\alpha}$ that give rise to the nonsymmetric and nonidempotent contributions. Substitution into Eq. (2.1) gives

$$F=F_0-\int \tilde{h}_{\lambda\beta}\sigma_{\lambda\beta}d^3\vec{r}, \quad (2.6)$$

where $\tilde{h}_{\lambda\beta}=-(\delta\mathcal{F}/\delta Q_{\mu\alpha})[(\delta_{\lambda\alpha}-Q_{\lambda\alpha})\delta_{\beta\mu}+(\delta_{\beta\alpha}-Q_{\beta\alpha})\delta_{\lambda\mu}]$. Since $\sigma_{\alpha\beta}$ is an unconstrained variation, $\tilde{h}_{\lambda\beta}$ is the thermodynamic force acting on $Q_{\lambda\beta}$. Here $\delta\mathcal{F}/\delta Q_{\lambda\beta}=-h_{\lambda\beta}$ is the molecular field of de Gennes [10]. The dynamical equation for $Q_{\lambda\beta}$, in the absence of flow, is given by the equilibrium between $\tilde{h}_{\lambda\beta}$ and the viscous force, that is,

$$\gamma\frac{\partial Q_{\lambda\beta}}{\partial t}=\tilde{h}_{\lambda\beta}, \quad (2.7)$$

where γ is a viscosity coefficient.

We restrict our analysis to consider the simple case of an equal elastic constant approximation ($K\equiv K_1=K_2=K_3$) for which $L_2=0$. In this case

$$h_{\alpha\mu}=L_1\partial_\omega\partial_\omega Q_{\alpha\mu}+\mathrm{Re}\{D^a_\alpha E^*_\mu\}, \quad (2.8)$$

where Re denotes the real part; thus the equation of motion is given by

$$\gamma\frac{\partial Q_{\lambda\beta}}{\partial t}=[(\delta_{\lambda\alpha}-Q_{\lambda\alpha})\delta_{\beta\mu}+(\delta_{\beta\alpha}-Q_{\beta\alpha})\delta_{\lambda\mu}] \times\{L_1\partial_\omega\partial_\omega Q_{\mu\alpha}+\mathrm{Re}(D^a_\mu E^*_\alpha)\}. \quad (2.9)$$

It is important to point out that Eq. (2.9) reduces to the corresponding equation for $\hat{n}$ [9] if we substitute $Q_{\mu\nu}=n_\mu n_\nu$. Following the usual procedure for decoupling Maxwell's equations, it is straightforward to show that the electric field propagating in a nonmagnetic nematic is governed by the equation

$$\partial_\mu\partial_\mu E_\lambda-\partial_\lambda\partial_\mu E_\mu+\mu_m\frac{\partial^2}{\partial t^2}(D_\lambda+Q_{\lambda\mu}D^a_\mu)=0 \quad (2.10)$$

in SI units. Here μ_m is the magnetic susceptibility and D_λ and D^a_λ are given by Eqs. (2.4). Notice that Eqs. (2.9) and (2.10) provide the complete general coupled dynamics of the nematic liquid crystal and electromagnetic field in the dyad representation.

III. WEAKLY NONLINEAR DYNAMICS

We consider the weakly nonlinear regime where the director is weakly distorted by the field. We consider a linear perturbation $q_{\lambda\beta}$ of a given initial stationary state $Q^0_{\lambda\beta}$, that is, $Q_{\lambda\beta}=Q^0_{\lambda\beta}+q_{\lambda\beta}$. Substitution of this expression into Eq. (2.9) yields

$$\gamma\frac{dq_{\lambda\beta}}{dt}=[(\delta_{\lambda\alpha}-Q^0_{\lambda\alpha})\delta_{\beta\mu}+(\delta_{\beta\alpha}-Q^0_{\beta\alpha})\delta_{\lambda\mu}] \times(L_1\partial_\sigma\partial_\sigma q_{\mu\alpha}+\mathrm{Re}\{D^a_\mu E^*_\alpha\}), \quad (3.1)$$

where we have kept only terms linear in $q_{\lambda\beta}$. In general, the unperturbed state $Q_{\mu\alpha}$ has to satisfy the condition

$$[(\delta_{\lambda\alpha}-Q^0_{\lambda\alpha})\delta_{\beta\mu}+(\delta_{\beta\alpha}-Q^0_{\beta\alpha})\delta_{\lambda\mu}]\partial_\sigma\partial_\sigma Q^0_{\mu\alpha}=0. \quad (3.2)$$

For simplicity, here we restrict our analysis to a configuration such that $\partial_\sigma\partial_\sigma Q^0_{\mu\alpha}=0$. We proceed to solve the linear equation for $q_{\lambda\beta}$ [Eq. (3.1)]. The Fourier transform of $q_{\lambda\beta}$ is

$$\tilde{q}_{\lambda\beta}(\vec{k},\omega)=\int e^{\vec{k}\cdot\vec{r}-i\omega t}q_{\lambda\beta}(\vec{r},t)d^3x\,dt. \quad (3.3)$$

In terms of this, Eq. (3.1) gives

$$\{(i\omega\gamma+L_1\vec{k}\cdot\vec{k})\delta_{\beta\alpha}-L_1\vec{k}\cdot\vec{k}Q^0_{\beta\alpha}\}Q^0_{\mu\gamma}\tilde{q}_{\mu\alpha} =Q^0_{\mu\gamma}[\delta_{\beta\alpha}-Q^0_{\beta\alpha}]\mathrm{Re}\{\tilde{D}^a_\alpha\circ\tilde{E}^*_\mu(\vec{k},\omega)\}, \quad (3.4)$$

where we have multiplied through by $Q^0_{\lambda\gamma}$ and used its idempotent property $Q^0_{\lambda\gamma}Q^0_{\lambda\alpha}=Q^0_{\gamma\alpha}$. Here $\tilde{E}_\alpha(\vec{k},\omega)$ and $\tilde{D}_\mu(\vec{k},\omega)$ denote the Fourier transforms of the displacement and the electric field vector, respectively, and $\circ$ indicates the convolution defined by

$$\tilde{D}^a_\alpha\circ\tilde{E}^*_\mu(\vec{k},\omega) =\int d\vec{k}_1 d\omega_1 d\vec{k}_2 d\omega_2\tilde{D}^a_\alpha(\vec{k}_1,\omega_1)\tilde{E}^*_\mu(\vec{k}_2,\omega_2) \times\delta(\vec{k}_1+\vec{k}_2-\vec{k})\delta(\omega_1+\omega_2-\omega). \quad (3.5)$$

From Eq. (3.4) we find that

$$Q^0_{\mu\gamma}\tilde{q}_{\mu\alpha}=Q^0_{\mu\gamma}\frac{[\delta_{\beta\alpha}-Q^0_{\beta\alpha}]}{i\omega\gamma+L_1\vec{k}\cdot\vec{k}}\mathrm{Re}\{\tilde{D}^a_\mu\circ\tilde{E}^*_\alpha(\vec{k},\omega)\}. \quad (3.6)$$

A similar expression is obtained for $\tilde{q}_{\mu\delta}Q^0_{\delta\gamma}$. Solving these for $\tilde{q}_{\mu\delta}$,

$$\tilde{q}_{\mu\delta}=([\delta_{\delta\alpha}-Q^0_{\delta\alpha}]\delta_{\gamma\mu}+[\delta_{\mu\alpha}-Q^0_{\mu\alpha}]\delta_{\gamma\delta})\frac{\mathrm{Re}\{\tilde{D}^a_\gamma\circ\tilde{E}^*_\alpha(\vec{k},\omega)\}}{i\omega\gamma+L_1\vec{k}\cdot\vec{k}}. \tag{3.7}$$

As expected, $\tilde{q}_{\mu\delta}$ is symmetric and $\tilde{q}_{\mu\delta}Q^0_{\mu\delta}=0$. Finally, by taking the Fourier transform of Eq. (2.10), it follows that

$$k_\lambda k_\lambda E_\mu - k_\mu k_\lambda E_\lambda + \mu_m\omega^2(\tilde{D}_\mu+\tilde{Q}_{\mu\lambda}\circ\tilde{D}^a_\lambda)=0, \tag{3.8}$$

where $\tilde{Q}_{\mu\delta}=\tilde{Q}^0_{\mu\delta}+\tilde{q}_{\mu\delta}$ and $\tilde{Q}^0_{\mu\delta}$ is the Fourier transform of $Q^0_{\mu\delta}$. Then, substitution of $\tilde{q}_{\mu\delta}$ from Eq. (3.7) into Eq. (3.8) yields

$$\begin{aligned}
&k_\lambda k_\lambda E_\mu - k_\mu k_\lambda E_\lambda + \mu_m\omega^2\Big(\epsilon_\perp(\vec{k},\omega)\tilde{E}_\mu(\vec{k},\omega)\\
&\quad+\tilde{Q}^0_{\mu\lambda}\epsilon_a(\vec{k},\omega)\tilde{E}_\lambda(\vec{k},\omega)+([\delta_{\delta\alpha}-Q^0_{\delta\alpha}]\delta_{\gamma\mu}\\
&\quad+[\delta_{\mu\alpha}-Q^0_{\mu\alpha}]\delta_{\gamma\delta})\int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\omega_1 d\omega_2 d\omega_3\\
&\quad\times\frac{\epsilon_a(\vec{k}_1,\omega_1)\epsilon_a(\vec{k}_3,\omega_3)}{i\gamma(\omega_1+\omega_2)+L_1|\vec{k}_1+\vec{k}_2|^2}\\
&\quad\times\mathrm{Re}\{\tilde{E}_\gamma(\vec{k}_1,\omega_1)\tilde{E}^*_\alpha(\vec{k}_2,\omega_2)\}\tilde{E}_\delta(\vec{k}_3,\omega_3)\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3\\
&\quad-\vec{k})\,\delta(\omega_1+\omega_2+\omega_3-\omega)\Big)=0,
\end{aligned} \tag{3.9}$$

which is a closed nonlinear equation, valid in the weakly nonlinear regime, for an arbitrary $\tilde{E}_\lambda(\vec{k},\omega)$ propagating in a nematic liquid crystal with an initially stationary configuration $Q^0_{\delta\alpha}$. Next we derive the amplitude equation for a wave packet from Eq. (3.9) by using a general operational procedure.

IV. AMPLITUDE EQUATION

Let us consider a wave packet of narrow bandwidth $\mu=\Delta\omega/\omega_0\ll 1$, central frequency ω_0, and wave vector $\vec{k}_0$, which travels in the nematic liquid crystal. We restrict our derivation to the case when the field is linearly polarized and propagates along the principal axis of the uniaxial stationary configuration $Q_{\delta\alpha}$. In a coordinate system where the z axis is parallel to the principal axis, $Q_{\delta\alpha}$ is diagonal. We look for a solution of Eq. (3.9) of the form

$$\tilde{E}_\mu=\mu[\tilde{A}(\vec{k},\omega)+\tilde{A}(\vec{k},-\omega)]x_\mu, \tag{4.1}$$

where x_μ is a unit vector perpendicular to z and $\tilde{A}(\vec{k},\omega)$ is assumed to have a pronounced peak at $\vec{k}_0,\omega_0$ so that $A(\vec{r},t)$ is a slowly varying function of space and time. In addition, we assume the following operational expressions for the frequency and wave-vector [11] components of the whole narrow wave packet:

$$\omega=\omega_0+\mu\frac{\partial}{\partial T}, \tag{4.2}$$

$$k_z=k_0+\mu\frac{\partial}{\partial Z}+\mu^2\frac{\partial}{\partial Z_2}+\cdots, \tag{4.3}$$

$$k_x=\mu\frac{\partial}{\partial X}, \tag{4.4}$$

$$k_y=\mu\frac{\partial}{\partial Y}, \tag{4.5}$$

where $t=\mu T$ is the time and $z=\mu Z_1=\mu^2 Z_2$ and $Z_2>Z_1$ are different longitudinal length scales that correspond to superior harmonic contributions [12]. The field amplitude and the wave-packet bandwidth are scaled by the same small parameter μ, which is the appropriate scaling for Kerr-like media. Substitution of Eq. (4.1) into Eq. (3.9) gives $k_x=0$ and

$$\begin{aligned}
&\mu[\mu_m\omega^2\epsilon(\vec{k},\omega)-k_y^2-k_z^2]\tilde{A}(\vec{k},\omega)\\
&\quad+2\mu^3[1-Q_{11}]\int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\omega_1 d\omega_2 d\omega_3\\
&\quad\times\frac{\epsilon_a(\vec{k}_1,\omega_1)\epsilon_a(\vec{k}_3,\omega_3)}{i\gamma(\omega_1+\omega_2)+L_1|\vec{k}_1+\vec{k}_2|^2}\\
&\quad\times\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3-\vec{k})\,\delta(\omega_1+\omega_2+\omega_3-\omega)\\
&\quad\times A(\vec{k}_1,\omega_1)A^*(\vec{k}_2,\omega_2)A(\vec{k}_3,\omega_3)=0.
\end{aligned} \tag{4.6}$$

We have introduced the abbreviation $\epsilon=\epsilon_\perp+\epsilon_a Q_{11}$. Note that if we solve this problem for the case where the director is parallel to the electric vector the nonlinear term vanishes, as expected; hence we consider only the case when the director and the field are orthogonal.

Substitution of Eqs. (4.2)–(4.5) into Eq. (3.9) and expanding in a Taylor series leads to a partial differential equation for $A(\vec{r},t)$. Since $A(\vec{k},\omega)$ varies rapidly near $\vec{k}_0,\omega_0$, we keep terms only up to third order in μ. Expanding the last term in Eq. (4.6) gives

$$\begin{aligned}
&\int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 d\omega_1 d\omega_2 d\omega_3\frac{\epsilon_a(\vec{k}_1,\omega_1)\epsilon_a(\vec{k}_3,\omega_3)}{i\gamma(\omega_1+\omega_2)+L_1|\vec{k}_1+\vec{k}_2|^2}\\
&\quad\times\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3-\vec{k})\,\delta(\omega_1+\omega_2+\omega_3-\omega)\\
&\quad\times A(\vec{k}_1,\omega_1)A^*(\vec{k}_2,\omega_2)A(\vec{k}_3,\omega_3)\\
&\approx\frac{\epsilon_a(\vec{k}_0,\omega_0)\epsilon_a(\vec{k}_0,\omega_0)}{i\gamma\omega_0+2L_1|\vec{k}_0|^2}A(\vec{r},t)A^*(\vec{r},t)A(\vec{r},t)\\
&\quad+\frac{\epsilon_a(\vec{k}_0,\omega_0)\epsilon_a(\vec{k}_0,-\omega_0)}{2L_1|\vec{k}_0|^2}A(\vec{r},t)A^*(\vec{r},t)A(\vec{r},t)\\
&\quad+O(\mu),
\end{aligned} \tag{4.7}$$

where we have chosen the coupling conditions given by $\vec{k}_1+\vec{k}_2=2\vec{k}_0$ and $\vec{k}_3=-\vec{k}_0$, which correspond to self-focusing

[11]. We have also taken $\epsilon(k_0,-\omega_0)=\epsilon(k_0,\omega)$, assuming absorption to be negligible. By grouping terms of the same order in μ we find

$$\mu: \quad -k_0^2+\mu_m\omega_0^2\epsilon(\omega_0)=0, \tag{4.8}$$

$$\mu^2: \quad 2ik_0\frac{\partial A}{\partial Z_1}+i\frac{\partial}{\partial\omega}(\mu_m\omega_0^2\epsilon)\frac{\partial A}{\partial T}=0, \tag{4.9}$$

$$\mu^3: \quad \frac{\partial^2 A}{\partial Y^2}+2ik_0\frac{\partial A}{\partial Z_2}+\frac{\partial^2 A}{\partial Z_1^2}-\frac{1}{2}\frac{\partial^2}{\partial\omega^2}(\mu_m\omega_0^2\epsilon)\frac{\partial^2 A}{\partial T^2} + \mu_m\omega_0^2\frac{\epsilon_a^2(\omega_0)|A|^2A}{2L_1k_0^2+i\omega_0}+\mu_m\omega_0^2\frac{\epsilon_a^2(\omega_0)|A|^2A}{2L_1k_0^2}=0. \tag{4.10}$$

Equation (4.8) is the linear dispersion relation and Eq. (4.9) shows that, to first order, a wave packet travels at the group velocity. Finally, substitution of Eq. (4.9) into Eq. (4.10) to eliminate Z_1 gives

$$\frac{\partial A}{\partial Z_2}-\frac{i}{2k_0}\frac{\partial^2 A}{\partial Y^2}+\frac{i}{2}\frac{\partial^2}{\partial\omega^2}\left(\frac{\omega_0\sqrt{\epsilon_\perp}}{c}\right)\frac{\partial^2 A}{\partial T^2}-i\frac{\epsilon_a^2(\omega_0)}{\epsilon_\perp}\frac{|A|^2A}{2L_1k_0} -i\frac{k_0}{\omega_0\gamma}\frac{\epsilon_a^2(\omega_0)}{\epsilon_\perp}\frac{(\beta-i)|A|^2A}{\beta^2+1}=0, \tag{4.11}$$

where $\beta=2L_1k_0^2/\omega_0\gamma$. This is the NLS equation with an additional complex quadratic term whose real part is proportional to the viscosity γ. This real part takes into account dissipation by the nematic liquid crystal associated with the reorientation process. It is interesting to note that Eq. (4.11) reduces to an equation of the form used by McLaughlin *et al.*, which describes the finer scale of undulation and filamentation [6] for the time-independent $\partial A/\partial t=0$ and lossless $\gamma=0$ case. Equation (4.11) takes into account the dispersion, self-focusing, and diffraction as well as dissipation by the nematic liquid crystal.

It is straightforward to show that the last term of Eq. (4.11) is small by substituting numerical values for a typical nematic liquid crystal: $L_1=10^{-11}N$, $\gamma=10^{-3}\,\mathrm{K\,g\,s^{-1}\,m^{-1}}$, and optical frequency $\omega_0=3.8\times10^{15}$ rad/s. This gives the value $\beta=1.3\times10^{-9}$; hence the fifth term is approximately real and much smaller than the fourth term. From the fourth term we obtain the nonlinear index n_2^{nem} as [11]

$$n_2^{nem}=\frac{\epsilon_a^2}{\epsilon_\perp k_0^2L_1}. \tag{4.12}$$

For typical values $\epsilon_a=0.64\epsilon_0$, $\epsilon_\perp=2.25\epsilon_0$, where ϵ_0 is the permittivity of the vacuum, and the parameter values given above $n_2^{nem}=1.0\times10^{-21}$ (km/V)^2, which is seven orders of magnitude larger than $n_2^{SiO_2}=1.2\times10^{-28}$ (km/V)^2. This is the giant nonlinearity expected for a liquid crystal [1]. Another physical quantity that can be estimated is the coefficient of the third term of Eq. (4.11), the dispersion. Using

$$n_0=1+n_0^0+\frac{g_0^1}{\omega_1^2-\omega^2}+\frac{g_0^2}{\omega_2^2-\omega^2}, \tag{4.13}$$

where $n_0^0=0.4136$, $\omega_1=8.9\times10^{15}$ rad/s, $\omega_2=6.68\times10^{15}$ rad/s, $g_0^1=4.8\times10^{30}$ (rad/s)^2, and $g_0^2=1.66\times10^{30}$ (rad/s)^2 for 4-pentyl-4′-cyanobiphenyl (5CB) from Ref. [14], we find $[d^2(kn_0)/d\omega^2(\mathrm{5CB})\approx1.1\times10^{-4}$ ps^2/km]. Thus the width of a picosecond pulse traveling in 5CB in the linear regime is doubled in a distance of 0.1 m, while in glass $[(d^2kn_o/d\omega^2)$ $(SiO_2)=1.8$ ps^2/km] it is doubled in a distance of 0.5 km. This is consistent with liquids being considerably more dispersive than solids.

V. SOLITON: ONE-DIMENSIONAL SOLUTION

We next restrict ourselves to the one-dimensional (1D) case. This means that we ignore the diffraction term $\partial^2A/\partial Y^2$. Equation (4.11) can then be rewritten in terms of dimensionless variables as

$$\frac{\partial\bar{A}}{\partial\bar{Z}}+i\frac{\partial^2\bar{A}}{\partial\bar{T}^2}-i|\bar{A}|^2\bar{A}-\beta|\bar{A}|^2\bar{A}=0, \tag{5.1}$$

where $\bar{A}\equiv A/A_0$, $\bar{Z}\equiv Z_2/Z_0$, and $\bar{T}\equiv T/T_0$. Here A_0 is the energy density of the optical pulse and Z_0,T_0 are the length and time scales given by

$$Z_0=\frac{2\epsilon_\perp k_0L_1}{A_0^2\epsilon_a^2} \tag{5.2}$$

and

$$T_0^2=\frac{2\epsilon_\perp k_0L_1}{\epsilon_a^2A_0^2}\frac{\partial^2\left(\frac{\omega_0}{c}\sqrt{\epsilon_\perp}\right)}{\partial^2\omega^2}. \tag{5.3}$$

Since β is small in Eq. (5.1), we consider the last term as a perturbation of the NLS equation, whose soliton type solution is given by [11]

$$A=2\eta\,\mathrm{sech}[\bar{T}-\bar{Z}(dk/d\omega)Z_0/T_0] \times\exp[ik(\omega_0)Z_0\bar{Z}-i\omega_0T_0\bar{T}], \tag{5.4}$$

where $\eta=A_0$. This kind of perturbative term has been treated previously [13] and it is found that it only modifies η in Eq. (5.4) as

$$\frac{d\eta}{d\bar{Z}}=-\frac{8}{3}\beta\eta^3. \tag{5.5}$$

By imposing the initial condition $\eta(\bar{Z}=0)=1$ leads to

$$\eta(\bar{Z})=\frac{1}{\sqrt{1+16\beta\bar{Z}/3}}. \tag{5.6}$$

This expression shows that dissipation due to reorientation makes the soliton amplitude η decrease with the distance. η falls half its initial amplitude A_0 when the soliton has traveled the distance $\overline{Z_a}=9/16\beta$.

We estimate the length and time scales of this pulse given by Eqs. (5.2) and (5.3). For a 500-mW laser at $\lambda=0.5\,\mu$m to a beam waist of 10 μm, the field amplitude is $A_0^2=1.9\times10^6$ V/m. Then, by using the material values given above, this leads to the spatial and temporal scales for the pulse $Z_0=4.2\times10^{-5}$ m and $T_0=0.21\times10^{-11}$ s.

From Eq. (5.4) we find that the soliton propagates with the speed $\overline{v}=v/c$,

$$\overline{v}=\frac{d\omega}{dk}\frac{Z_0}{T_0}=\frac{n}{c}\frac{1}{A_0}\sqrt{\frac{\lambda_0}{2\pi n_2 d^2(kn_0)/d\omega^2}} \tag{5.7}$$

and for our values $\overline{v}^{nem}=0.1$, which is one order of magnitude smaller than the speed of light c in vacuum and has the same as $\overline{v}^{SiO_2}=2.5\times10^{-1}$. The difference between $\overline{v}^{nem}$ and $\overline{v}^{SiO_2}$ comes from the product $n_2 d^2(kn_0)/d\omega^2$ in Eq. (5.7), which measures the balance between nonlinearity and dispersion.

Finally, the characteristic distance over which the soliton loses half of its initial amplitude A_0 is given by $z_d=Z_0 Z_a$, which leads to $z_d^{nem}=12$ km. The corresponding distance for SiO_2 is $z_d^{SiO_2}=50$ km, which is larger than z_d^{nem} due to losses associated with the reorientation of the nematic liquid crystal. It is important to note that $z_d^{SiO_2}$ was calculated [15] assuming a linear lossy perturbative term of the form $-\gamma A$, in contrast with the nonlinear term in Eq. (5.1). It is interesting to point out that some other solutions of the standard NLS equation that are not solitonlike, but are instead wavelike coherent structures also may be solutions [16] of the perturbed equation (5.1).

VI. SUMMARY

We have derived a nonlocal and retarded equation for the electric field in the bulk nematic liquid crystal in the weakly nonlinear regime. From this we derived a nonlinear Schrödinger equation for the amplitude that takes into account self-focusing, dispersion, diffraction, and dissipation in the nematic liquid crystal. We have shown the existence of a solitonlike (1D) solution and we estimated its speed, time and length scales and absorption length.

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